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On Some Definition of Physical State

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We propose to describe physical systems (quantum or classical) with the aid of structures called information systems. We argue that in each theoretical description of the phenomenology of the preparation process one can identify the set of all (pure) states of a physical system with the set of (total) elements of some information system. Next we give some consequences of the above arguments.

1. INTRODUCTION

In this paper we propose to describe physical systems (quantum or classical) with the aid of structures called information systems. Information systems are used already in the theory of domains for denotational semantics (Scott, 1982). The mathematical structure of an information system follows from some basic notions that we have about physical reality and properties of truth.

We are aware of the fact that the notion of a physical system is often quite opaque and misleading and may cause many problems related, e.g., with the EPR-type experiments. Generally speaking by physical system we will understand here an object that we single out from the universe and whose interaction with the rest of the universe can be—to sufficient degree of approximation—neglected. A concrete theory describes and studies these more or less idealized objects by means of a suitable mathematical structure.

Our approach is concerned with individual physical systems and can be employed for classical and quantum systems as well. Of course one can always go on to a statistical interpretation, forming the appropriate ensemble of uncorrelated replicas of individual systems.

We will assume that in each moment of time every individual system exists by itself and is in a definite internal state embodying the characteristics

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of definite properties. A property of a system is comprehended here in the sense of an actual property or what Einstein called an "element of physical reality."

As Gisin says: "an element of reality—or equivalently an 'actual property' in Piron's terminology—is a property of the physical system that can be tested and that is such that if we would actually perform a test, the positive result will always come out. Consequently, the elements of reality only depend on the system (and not on the measuring apparatus): They are in some way engraved in the system."

They are engraved in the system through the method of preparation of the state of the system and they should be consistent with the nature of the system as well.

If the electron has been prepared in an eigenstate of the momentum operator with eigenvalue p then necessarily, if we make the measurement of the momentum, the result will be p. This interpretation is the realistic one in the strong sense and satisfies all the requirements of the well-known Einstein criticism of the usual statistical interpretation (Aerts, 1979). Internal states of the system, and so the actual properties which the states *have*, depend on preparation procedures. Therefore every preparation procedure states some family of (actual) properties of a physical state that was prepared according to it. It results from this that a property that, in a given theory describing our system, one can attribute to a state independently of whether a registration of the result of a measurement of that property later takes place, depends on the data concerning the method of preparation of the system and on assumptions and rules of inference standing in the theory.

This is, indeed, precisely the role demanded of a (pure) state specification: namely, to determine maximally informative (maximally precise) *consistent* description of a physical system.

2. THE SET OF STATES OF A PHYSICAL SYSTEM

Definition (Scott, 1982). An *information system* is a structure:

$$(D, \Delta, \mathrm{Con}, \vdash)$$

where D is a set, where Δ is a distinguished member of D (the *least informative* member), where Con is a set of finite subsets of D (the *consistent* sets), and where $\vdash$ is a binary relation between members of Con and members of D (the *entailment* relation).

Concerning Con, the following axioms must be satisfied for all finite subsets u, $v \subseteq D$:

(1) $u \in \text{Con}$, whenever $u \subseteq v \in \text{Con}$

(2) $\{X\} \in \text{Con}$, whenever $X \in D$

(3) $u \cup \{X\} \in \text{Con}$, whenever $u \vdash X$

Concerning $\vdash$, the following axioms must be satisfied for all u, $v \in \text{Con}$, and all $X \in D$:

(4) $u \vdash \Delta$

(5) $u \vdash X$, whenever $X \in u$

(6) if $v \vdash Y$ for all $Y \in u$ and $u \vdash X$, then $v \vdash X$

In our case we may think of the members of D as of (actual) properties which our individual physical system may have.

Δ is the trivial property engraved in each state of the system (e.g., the property that the system under consideration is present). If $u \in \text{Con}$ is false, then the properties from u are never simultaneously actual.

One can easily give a vast amount of consistent or nonconsistent sets of properties of physical systems, quantum or classical as well.

Example. Consider a piece of lead. Then the properties "the temperature of the piece is 1000°C" and "the piece has shape of a cube with edge 1 cm" are obviously inconsistent. The structure of the set Con depends of course on a theory which describes our system and may change with the development of the theory.

Some of the properties may be mutually dependent. The entailment relation $\vdash$ for D should be constructed so as to respect the intended meaning of the properties of the system. So $\vdash$ is interpretated here in the sense of a semantical relation of implication, i.e., the minimal assumptions on the relation $\vdash$ should give it the required property that $v \vdash X$ iff whenever the system has the properties belonging to u then it has property X.

The entailment relation is always relative to a class of models (worlds, states, circumstances, situations etc.). We have examples of appropriate relations of (semantical) entailment in propositional calculus (consequence relations *à la* Tarski), in quantum "logic" (defined by means of the partial ordering of "quantum lattice"), in modal realizations of quantum "logic" (Dishkant, 1972, Dalla Chiara, 1977; Goldblatt, 1974).

Sometimes the main part of the structure of an information system may lie in the set Con, sometimes in the entailment relation, and sometimes it is in the interplay between the two notions (Scott, 1982).

In most of the interpretations of "logic" of quantum systems one can easily point out the structure of an information system. The interpretations single out the main difference between classical and quantum theory in the structure of the set and/or in designation of the semantical entailment (Finkelstein, 1969). It seems reasonable to assume that to each physical system there is associated in information system $(D, \Delta, \vdash, \mathrm{Con})$ where the set D, entailment relation $\vdash$, the least informative member Δ, and the set Con depend on the nature of the system and on theory which describes it.

Now (see Introduction) we argue that the (partial) state of one given physical system at time t_0 is uniquely determined by the set of properties engraved in the system by preparation procedure and actual at that time t_0.

Respectively, the pure state may be identified with the collection of *all* actual properties of the system. This is exactly the definition of (pure) state given by Jauch and Piron (Jauch and Piron, 1969).

One can ask which subsets of properties can be taken to define the states of a system. It is rather reasonable to assume that the subsets should be consistent in themselves (because they describe real physical systems) and deductively closed: (entailment should be truth preserving).

To return to the Scott theory of domains we can easily see that strict notional counterpart of our conception of (pure) state is that of (total) element.

Definition (Scott, 1982). The *elements* of the information system $A = (D_A, \Delta_A, \mathrm{Con}_A, \vdash_A)$ are those subsets x of D_A where (1) all finite subsets of X are in Con_A, and (2) whenever $u \subseteq X$ and $u \vdash_A X$, then $X \in x$. We write $x \in |A|$ to mean x is an element of the system. An element that is not included in any strictly larger element in the set $|A|$ is called a *total* element; the set of total elements is denoted by Tot_A.

Therefore it seems reasonable to accept the following:

Assumption. In each theoretical description of the phenomenology of the preparation process one can identify the set of all (pure) states of a physical system with the set of (total) elements of some information system.

One can give many examples of (total) elements in various mathematical formalisms. In propositional calculus elements are called theories, in modal logic, possible worlds; in quantum "logic" the Jauch–Piron states are examples of total elements.

Because the elements (states) of an information system are introduced as sets, there is a natural relation of partial ordering under set-theoretic inclusion. $x \subseteq y$ means that each actual property of x is also a property of

y. Moreover because for two elements x, $y \in |A|$ their intersection $x \cap y$ is also an element, one can easily show (Scott, 1982) that $|A|$ is a (conditionally) complete inf semilattice.

Now we would like to enrich the structure of the set of states. Perhaps the most important thing to have in mind if one is about to make precise the concept of an approximate statement of the properties of an element of some structure is that one needs a *topology*, for a topology is the very mathematical tool to settle all matters in which our intuition works with the idea of an approximation.

We have a very natural notion of topology in the theory of information systems, the notion which we could easily adopt with obvious interpretation to the set of physical states.

Definition (Scott, 1982). Consider an information system A. For each $u \in \mathrm{Con}_A$, we define a corresponding neighborhood of $|A|$ by the equation

$$[u]_A = \{y \in |A| \colon u \subseteq y\}$$

The neighborhoods of an element x are all those sets $[u]_A$ where $u \in x$.

The topology in the set of elements $|A|$ is generated by so defined family of neighborhoods.

The above definition is entirely consistent with the intuitive notion of a neighborhood of an element (state).

A neighborhood of an element (state)$x \in |A|$, generated by a finite element $u \in \mathrm{Con}_A$, it is a set of all these elements states) y, which differ from the element x no more than u. The relation "differs no more than" is determined in terms of the relation of partial order Con_A in the set $|A|$, and in our case it is connected with amount of information about the elements (states).

It seems that in the framework of our model of the set of states of physical system, the above definition of topology is the most natural one.

Theorem (Scott, 1982). The space $\mathrm{Tot}_A \subseteq |A|$ with the induced topology is a totally disconnected, compact Hausdorff space.

The theorem is very important for further applications. for instance in the so-called "convex" approach to the foundations of quantum mechanics the main part is played by the convex set of all states of a physical system—the "statistical figure" (Mielnik, 1974). Mielnik in his papers (Mielnik, 1974, 1980) has given a general recipe of the construction of "statistical figure," but only "up to topological questions" (Haag and Bannier, 1978). In our paper (Posiewnik, 1984) we gave the topological details lacking in the papers of Mielnik and Haag and Bannier and we showed that the Mielnik construction is possible to perform in the case when the set of pure states is equipped with a compact topology.

In this instance the "statistical figure" is obtained as a compact convex set in locally convex Hausdorff topology. Then it is a good starting point for the construction of a statistical theory in which the mixtures of physical states are described in the terms of the Choquet theory (Alfsen, 1971). A detailed analysis of the problem can be found in Ref. 16 of Alfsen (1971).

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